JOURNAL OF ALGEBRA **104**, 57–60 (1986)

Indecomposable Endo-Permutation Modules over p-Groups

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Received June 14, 1985

DEDICATED TO PROFESSOR GREEN ON THE OCCASION OF HIS 60TH BIRTHDAY

We use results of Green to give a correct proof that the modules of the title are all absolutely indecomposable in the sense of Huppert. © 1986 Academic Press, Inc.

In the papers [3–6] Green invented several absolutely basic concepts, such as vertices, sources, and what are now known as Green correspondents of indecomposable group modules. He also gave many of the main theorems about them. These concepts and theorems are so fundamental in modern representation theory that using them is like breathing: you don't think about it, you just do it. Sometimes, however, this automatic application of Green's ideas can lead the unwary astray. It happened to us not so long ago in [1]. The resulting error is instructive.

We were working with a prime p, a finite p-group P and a valuation ring $\mathfrak{O}$ whose residue class field $\mathfrak{F}$ had characteristic p. We were studying lattices $\mathfrak{L}$ over the group algebra $\mathfrak{O}P$ of P with coefficients in $\mathfrak{O}$. In order to apply Green's theories, we assumed (in (1.1) of [1]) the following:

HYPOTHESIS. *If Q is any subgroup of P and $\mathfrak{K}$ is any indecomposable $\mathfrak{O}Q$-lattice, then the $\mathfrak{O}$-order* $\mathrm{End}_{\mathfrak{O}Q}(\mathfrak{K})$ *of all $\mathfrak{O}Q$-endomorphisms of $\mathfrak{K}$ is local in the sense that its factor $\mathfrak{F}$-algebra*:

$$\mathfrak{E}(\mathfrak{K}) = \mathrm{End}_{\mathfrak{O}Q}(\mathfrak{K})/\mathrm{J}(\mathrm{End}_{\mathfrak{O}Q}(\mathfrak{K})) \qquad (1)$$

modulo its Jacobson radical is a division algebra.

This implied the Krull–Schmidt Theorem for lattices over subgroups of P, and was enough to give us all of Green's results about vertices, sources

*This work was partially supported by a grant from the National Science Foundation.

0021-8693/86 $3.00

and Green correspondents of indecomposable $\mathfrak{O}P$-lattices. In particular, Green's Theorem 8 in [3] became, in our situation:

THEOREM 2 (Green). *If Q is a subgroup of P and $\mathfrak{K}$ is any absolutely indecomposable $\mathfrak{O}Q$-lattice, then the induced $\mathfrak{O}P$-lattice $\mathfrak{K}^P$ is also absolutely indecomposable.*

What we really used was the following easy consequence of Green's Theorem:

COROLLARY 3. *If $\mathfrak{L}$ is an indecomposable $\mathfrak{O}Q$-lattice having an absolutely indecomposable $\mathfrak{O}Q$-lattice $\mathfrak{K}$ as a source, for some vertex Q, then $\mathfrak{L}$ is absolutely indecomposable and isomorphic to $\mathfrak{K}^P$.*

Of course, this held since $\mathfrak{L}$ was a divisor (in the sense of Green [3], i.e., isomorphic to an $\mathfrak{O}P$-direct summand) of $\mathfrak{K}^P$, which was absolutely indecomposable by Theorem 2.

The objects of our study in [1] were the *endo-permutation $\mathfrak{O}P$-modules*, i.e., those $\mathfrak{O}P$-lattices $\mathfrak{L}$ whose orders $\mathrm{End}_{\mathfrak{O}}(\mathfrak{L})$ of all $\mathfrak{O}$-endomorphisms were, under the natural conjugation action of P, permutation $\mathfrak{O}P$-modules in the sense of section 2.3 of Green's paper [4]. After considerable work we had reached Lemma 6.4 of [1], which can be stated as

LEMMA 4. *If $\mathfrak{L}$ is an indecomposable endo-permutation $\mathfrak{O}P$-module with vertex P, then $\mathfrak{L}$ is absolutely indecomposable.*

We then applied Green's Theorem 2 to remove the hypothesis about the vertex of $\mathfrak{L}$ and to prove, in Theorem 6.6 of [1], the:

THEOREM 5. *If $\mathfrak{L}$ is any indecomposable endo-permutation $\mathfrak{O}P$-module, then $\mathfrak{L}$ is absolutely indecomposable.*

To do so, we chose a vertex Q and a corresponding source $\mathfrak{K}$ for $\mathfrak{L}$. Since $\mathfrak{K}$ was a divisor of the restriction $\mathfrak{L}_Q$ of $\mathfrak{L}$ to an $\mathfrak{O}Q$-lattice, it was easily seen to be an endo-permutation $\mathfrak{O}Q$-module. Since it was indecomposable with vertex Q, it was absolutely indecomposable by Lemma 4. So $\mathfrak{L}$ was absolutely indecomposable by Corollary 3, and Theorem 5 was proved.

The above proof is a perfectly straightforward, almost automatic, application of Green's Theorem 2 to get from Lemma 4 to Theorem 5. Nevertheless, it hides a subtle error. The problem is that the "absolute indecomposability" in Green's Theorem 2 is not the same as that in Lemma 4, which, by implication, should be that of Theorem 5. Green used the natural definition in [3] and [4], under which an $\mathfrak{O}P$-lattice is absolutely indecomposable if it remains indecomposable under any extension of the ground ring, i.e., if the $\mathfrak{O}'P$-lattice $\mathfrak{O}' \otimes_{\mathfrak{O}} \mathfrak{L}$ is indecomposable whenever $\mathfrak{O}'$ is any valuation ring having $\mathfrak{O}$ as a subring such that $J(\mathfrak{O}') \cap \mathfrak{O} = J(\mathfrak{O})$. This is easily translated into a property of the 'residual endomorphism algebra' $\mathfrak{E}(\mathfrak{L})$ defined in (1). If then becomes:

GREEN'S DEFINITION 6. An $\mathfrak{O}P$-lattice $\mathfrak{L}$ is *absolutely indecomposable* if $\mathfrak{E}(\mathfrak{L})$ is a purely inseparable extension field of $\mathfrak{F}$.

The definition used in Lemma 4 was due to Huppert in [7]. It is:

HUPPERT'S DEFINITION 7. An $\mathfrak{O}P$-lattice $\mathfrak{L}$ is *absolutely indecomposable* if $\mathfrak{E}(\mathfrak{L})$ is isomorphic to $\mathfrak{F}$ as an $\mathfrak{F}$-algebra.

This was made quite explicit in Lemma 6.4 of [1]. Indeed, no other definition of absolute indecomposability was mentioned in [1]. So the reader had every right to expect that the absolute indecomposability in Theorem 5 was Huppert's. However, Green's Theorem 2 is simply false for Huppert's definition. There are easy examples where, in the notation of that theorem, $\mathfrak{K}$ satisfies Huppert's criterion and $\mathfrak{K}^P$ does not. So the above proof of Theorem 5 only shows that it holds using absolute indecomposability in the sense of Green.

We can't go about misleading readers like that. So we shall make amends by proving Theorem 5 in the strong form which was originally intended, i.e., by proving

THEOREM 8. *If $\mathfrak{L}$ is an indecomposable endo-permutation $\mathfrak{O}P$-module, then $\mathfrak{L}$ is absolutely indecomposable in the sense of Huppert's Definition* 7.

Proof. Let Q be a vertex of $\mathfrak{L}$ and H be its normalizer $N_P(Q)$ in P. By Theorem 2 of Green's paper [5], there is an indecomposable $\mathfrak{O}H$-lattice $\mathfrak{M}$ with vertex Q such that $\mathfrak{M}$ divides $\mathfrak{L}_Q$. Indeed, these conditions determine the *Green correspondent* $\mathfrak{M}$ of $\mathfrak{L}$ to within isomorphism. A result of Feit (Theorem 1(iv) of [2]) and Green (Theorem 4.1 of [6]) tells us that $\mathfrak{E}(\mathfrak{L})$ is isomorphic to the $\mathfrak{F}$-algebra $\mathfrak{E}(\mathfrak{M})$. So it suffices to prove that $\mathfrak{M}$ is absolutely indecomposable in Huppert's sense.

Some indecomposable $\mathfrak{O}Q$-lattice $\mathfrak{K}$ with vertex Q is a source of both $\mathfrak{M}$ and $\mathfrak{L}$. Since $\mathfrak{L}$ is an endo-permutation module, so is the divisor $\mathfrak{K}$ of $\mathfrak{L}_Q$. Hence $\mathfrak{K}$ is absolutely indecomposable (in Huppert's sense) by Lemma 4. Now Corollary 3 tells us that $\mathfrak{K}^H$ is indecomposable, and hence is isomorphic to its divisor $\mathfrak{M}$. So we may as well assume that:

$$\mathfrak{M} = \mathfrak{K}^H. \tag{9}$$

Proposition 6.2 and its Corollary 6.3 in [1] imply that $\bar{\mathfrak{M}} = \mathfrak{F} \otimes_{\mathfrak{O}} \mathfrak{M}$ is an indecomposable endo-permutation $\mathfrak{F}H$-module satisfying:

$$\mathfrak{E}(\mathfrak{M}) \simeq \mathfrak{E}(\bar{\mathfrak{M}}) \qquad (\textit{as } \mathfrak{F}\textit{-algebras}).$$

So it suffices to show that $\mathfrak{M}$ is absolutely indecomposable in Huppert's sense.

Clearly (9) implies that $\mathfrak{M}$ is induced from the indecomposable endo-permutation $\mathfrak{F}Q$-module $\bar{\mathfrak{K}} = \mathfrak{F} \otimes_{\mathfrak{O}} \mathfrak{K}$, which has vertex Q as in the first lines of the proof of Lemma 6.4 in [1]. Proposition 5.32 of [1] now says that the endo-permutation $\mathfrak{F}(H/Q)$-module $\mathfrak{M}/Q$, defined in Theorem 4.15 of [1], is isomorphic to $(\bar{\mathfrak{K}}/Q)^{H/Q}$. By Proposition 5.7 of [1] the $\mathfrak{F}(Q/Q)$-module $\bar{\mathfrak{K}}/Q$ is indecomposable, and hence is isomorphic to the one-dimensional trivial module. It follows that $\mathfrak{M}/Q$ is isomorphic to the regular $\mathfrak{F}(H/Q)$-module, which is absolutely indecomposable in Huppert's sense since H/Q is a p-group and $\mathfrak{F}$ has characteristic p. This and (5.2) of [1] imply that:

$$\mathfrak{E}(\mathfrak{M}) \simeq \mathfrak{E}(\mathfrak{M}/Q) \simeq \mathfrak{F} \qquad (\textit{as } \mathfrak{F}\textit{-algebras}).$$

So $\mathfrak{M}$ is absolutely indecomposable in Huppert's sense, and the theorem is proved.

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